DIURNAL TEMPERATURE WAVE

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Submitted in partial fulfillment of
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ABSTRACT

The partial differential equation for heat diffusion is numerically integrated by the Runga-Kutta method. Solutions are obtained for the diurnal temperature variation with a bounded coefficient of eddy diffusivity which varies with the lapse rate and with height. The surface wave is represented by the sum of a diurnal and a semidiurnal harmonic wave. The results may be interpreted to apply over a fairly broad range of diffusivity with height. With appropriate choices of the various parameters, reasonably good agreement is obtained between theoretical and observational values of amplitude and phase lag as functions of height and time.

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1. Introduction.

Many efforts have been made by meteorologists to solve the heat diffusion problem in the atmosphere, in particular the diurnal temperature wave has received much attention. In every attempt the concept of eddy diffusivity has been retained. the various functional forms of eddy diffusivity give different results. Most analyses are based upon the solution of the Taylor heat diffusion equation, which states that the turbulent flux of heat is proportional to the gradient of potential temperature, Namely:

$$\frac{\partial \theta}{\partial t} = \frac{\partial}{\partial z}\left(K \frac{\partial \theta}{\partial z}\right) \tag{1}$$

Here t represents the time, z the height, and $\theta(z, t)$ the potential temperature deviation from a constant (mean) value.

The simplest approach is based upon the assumption of a constant diffusion coefficient K. According to this solution the amplitude of the diurnal temperature wave decreases exponentially and phase increases linearly with height. For a variable K several solutions have been presented. Sutton (1) in his comprehensive survey of daily temperature variation has considered $K = K_1 z^m$, where m is the stability parameter having magnitude $0 \leq m \leq 1$. His final solution is in terms of Bessel functions. It may be inferred from his results that at sufficiently great heights, the diurnal variation becomes periodic with a phase lag which is proportional to $z^{1 - m/2}$, so that the exponent of z is now as low as 0.5 (for m=1). However, agreement with observation is not as good as might be expected in view of the additional complexity of the solution.

Best and Kohler (2) have made a similar approach to the problem. Haurwitz (3) has considered K as: $K = Ko + K_1 z$, where Ko and K_1 are constants. By proper selection of Ko and K_1, Haurwitz obtains good agreement with data taken at Leafield, England, up to 10 meters. In his results, the amplitude falls off more rapidly at low levels than in K Constant Theory. However, it is greater at high levels where K is larger. The phase lag is larger near the ground where K is small but becomes smaller at higher levels where K is larger. Poppendiek (4) has obtained a solution, for K varying sinusoidally with time and linearly with height. The solution is very complicated, consisting of an infinite trigonometric series having coefficients determined by difficult integrations of functions of the real and imaginary parts of Hankel functions of complex argument. No numerical values are given, and the results are not suitable for practical purposes. Even for the case of K independent of time but varying as a power of height, the solution is in terms of Bessel functions of order depending on this power. Staley (5) has investigated the problem with K increasing with height but bounded. In several respects the results show better agreement with observations than previous solutions for the coefficient unbounded. A numerical solution with a diffusion coefficient which varies periodically with time and exponentially with height has been obtained by Haltiner (7). In general the functional forms of the diffusion coefficient have not included such parameters as the roughness coefficient, gradient wind, lapse rate etc., which are generally believed to be closely

related to the diffusion process. It is the purpose of this investigation to attempt to determine a diffusion coefficient in which the time variation is related to the stability. Briefly, the present analysis consists of the selection of proper functional form for the coefficient of eddy diffusivity from observed data, followed by numerical solution of the Taylor heat diffusion equation for this value of K.

2. Heat Diffusivity Coefficient.

From physical considerations, there is no reason to expect an ever increasing K. Hence a bounded function of K appears to be a logical choice. Also, many studies on the variation of K with various meteorological parameters give clues to the formation of a suitable function. Sutton (1) has remarked that " the relatively small change of amplitude of the diurnal temperature wave above 10 meters in summer compared with that in winter indicates that a more effective mechanism for the upward transfer of heat is at work in the summer than in the winter". Also Haltiner (7) suggests that K should have diurnal variation.

It is possible that both the seasonal and diurnal variation in diffusion may reasonably be accounted for by making K a suitable function of lapse rate. Moreover from the observations made at the micrometeorological tower at the University of Washington, Fig. (1), which shows K as a function of time at three heights, it is evident that K at a given level increases rapidly by one or more orders of magnitude as the lapse rate changes from sub-adiabatic to super-adiabatic after sunrise. In the evening when the lapse rate changes from super-adiabatic to slightly sub-adiabatic K drops rapidly by one or more order of magnitude at any given level. Further from Fig.(2), which shows K as a function of lapse rate, it is evident that at lower levels the variation in K is less than at higher levels. On further investigation of the data collected at the University of Texas micrometeorological tower (see Table 1) it has been found that the variation in K at the lower levels is an order of magnitude (factor of ten) while

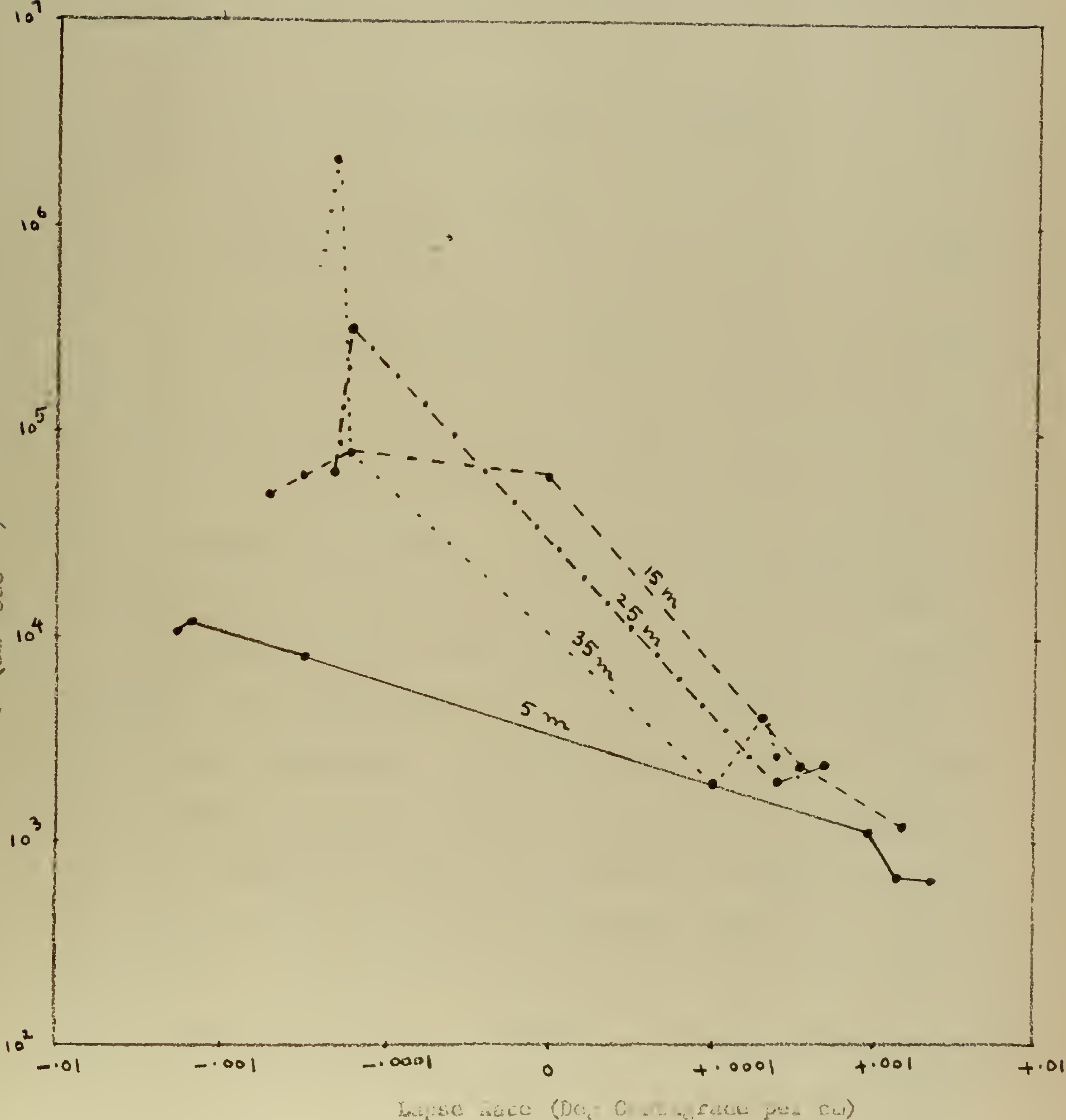
Fig. 1

Coefficient of eddy conductivity as a function of time at different heights; a: 0.75 m at University of Washington micro-meteorological mast and tower, represents mean over 23 days of Aug. 1950, chosen for having most nearly sinusoidal temperature variations; b and c : for 15 and 35 m, respectively (after Jehn and Gerherdt, 1950) , 3 - 4 Aug 1948, at Manor Texas. Dashed lines represent unreliability. All values obtained by methods based on energy continuity at earths surface.

Table 1

	5 (Meters)			15 (Meters)		
Time	$-\partial\theta/\partial z$	K_{Obs}	K_{Cal}	$-\partial\theta/\partial z$	K_{Obs}	K_{Cal}
1200	-.0025	1.2×10^4	6.1×10^4	-.0005	6.4×10^4	1.8×10^5
1600	-.0020	1.3×10^4	6.0×10^4	-.0007	5.4×10^4	2.0×10^5
2000	-.0015	7×10^2	5.0×10^4	-.0005	1.4×10^3	1.2×10^5
2400	-.0035	6.5×10^2	4.5×10^4	-.0005	2.5×10^3	1.2×10^5
0400	-.0010	1.2×10^3	5.2×10^4	-.0013	1.6×10^3	6.4×10^4
0800	-.0005	8×10^3	5.4×10^4	0	6.0×10^4	1.6×10^5
	25 (Meters)			35 (Meters)		
Time	$-\partial\theta/\partial z$	K_{Obs}	K_{Cal}	$-\partial\theta/\partial z$	K_{Obs}	K_{Cal}
1200	-.0002	2×10^5	2.6×10^5	-.0004	6.5×10^5	5.0×10^5
1600	-.0002	3×10^5	2.6×10^5	-.0003	1.2×10^6	4.4×10^5
2000	-.0004	2×10^3	1.0×10^5	-.0001	2.0×10^3	1.9×10^5
2400	-.0004	3×10^3	1.0×10^5	-.0004	2.7×10^3	5.1×10^3
0400	-.0007	2.5×10^3	2.6×10^4	-.0003	4.0×10^3	6.8×10^4
0800	-.0003	6.5×10^4	2.8×10^5	-.0002	8.0×10^4	3.8×10^5

FIG. 2



at higher levels K varies by two orders of magnitude. Therefore any function which represents K should show the above characteristics. As a preliminary study the following functions have been investigated:

$$K = a_1\left[1 - (a_2 + a_3 z)\frac{\partial\theta}{\partial z}\right]\left[1 - a_4 e^{-a_5 z}\right] \tag{2}$$

$$K = a_1\left[1 - a_2 e^{-a_3 z^2 \frac{\partial\theta}{\partial z}}\right]^2 \tag{3}$$

$$K = a_1\left(1 - a_2\frac{\partial\theta}{\partial z}\right) e^{-a_3 z^2 \frac{\partial\theta}{\partial z}} \tag{4}$$

$$K = a_1\left[1 - a_2 z^2 \frac{\partial\theta}{\partial z}\right]\left[1 - a_3 e^{-a_4 z}\right] \tag{5}$$

Here z represents the height, θ is the potential temperature deviation from mean value and a_1, a_2, a_3, a_4, a_5 are contants.

In equation (2) the second factor $(1 - a_4 e^{-a_5 z})$ is obviously bounded; and the other factor $(1-(a_2+a_3 z)\frac{\partial\theta}{\partial z})$ changes K as lapse rate changes. Furthermore, the variable z is also involved in this factor. This is necessary, because at great heights the change in lapse rate becomes very small and the factor z is needed to amplify the effect of lapse rate at these elevations. When $z = 0$, $\frac{\partial\theta}{\partial z}$ is maximum, then K reduces to $K = (a_1 - a_2\frac{\partial\theta}{\partial z})(1 - a_4)$ for $z = \infty$, $\frac{\partial\theta}{\partial z} = 0$ & $K = a_1$ In quality this equation possesses all characteristics mentioned above but the quantitative agreement with the observed data is not sufficiently good. In the second representation (3) K is bounded and it increases with height to a certain level which may be determined by the proper selection of constants. There is an appreciable

difference in day-time and night-time, but the night-time values are lower at high elevations than at lower levels. This function is very sensitive to changes of $\frac{\partial\theta}{\partial z}$ and in magnitude of z. The square of the quantity was intended to avoid negative values of K.

Form (4) is somewhat simpler than equation (3). In this equation also, K is bounded and has much the same characteristics as equation (3), but the variation in the value of K when lapse rate changes sign is less than the former. Diffusion coefficient (5) gives a reasonably good fit with the observed data although the night-time values are higher than observed. The difference between night and day-time values is correctly maintained.

For the present analysis, equation (5) has been selected as the best representation for K of this group. The following numerical values are chosen as being representative:

$$a_1 = 5 \times 10^5 \ \text{cm}^2 \ \text{sec}^{-1} \ ; \quad a_2 = .2 \times 10^{-3}$$

$$a_3 = .99 \qquad ; \quad a_4 = .2 \times 10^{-3}$$

The choice of a_3 gives a hundred-fold variation of K with height. The value of a_1 controls the overall magnitude of K:

$$\text{For } z = 0 \ , \ K \quad 5 \times 10^3 \ \text{cm}^2 \ \text{sec}^{-1}$$

$$\text{For } z = \infty \ , \ K \quad 5 \times 10^5 \ \text{cm}^2 \ \text{sec}^{-1}$$

The value of a_2 always keeps the value of K positive by limiting the value of $a_2 z^2 \partial\theta/\partial z < 1$ for all elevations and typical lapse rates. The parameter a_4 controls the rate of decrease of K with height.

3. Solution of Taylor Heat Diffusion Equation .

The associated boundary conditions of the heat diffusion equation (1) are:

$$\theta = 0 \quad \text{for } z = \infty \text{ and all } t \tag{6}$$

$$\theta = \theta_0(t) \quad \text{for } z = 0 \tag{7}$$

In the analysis done by Haltiner (7) $\theta_0(t)$ has been represented by two trigonometric terms which is normally a reasonable good approximation to the observed data. Thus the following form is assumed for:

$$\theta_0 = a_5 (\sin \omega t + c \sin 2 \omega t) \tag{8}$$

Here $\omega = \frac{2\pi}{86400} \sec^{-1}$; $a_5 = .087$ which for convenience keeps the magnitude of $\theta_0 \lesssim \cdot 1$; $c = 0.3$

In order to scale the equation for computational purposes, let

$$t = 3600\tau \text{ and } z = 100 q \sigma \tag{9}$$

Here q is a constant, while σ and τ are the new variables, with t in seconds, the units of τ are hours. For z in cm, σ will be units of q-meters, ie in meters when q = 1, in 2 meters unit when q = 2, etc., with the transformation (9) equations (1) and (4) become:

$$\frac{\partial \theta}{\partial \tau} = \frac{\cdot 36}{q^2} \left[K \frac{\partial^2 \theta}{\partial \sigma^2} + \frac{\partial \theta}{\partial \sigma} \frac{\partial K}{\partial \sigma} \right] \tag{10}$$

$$K = a_1 \left[1 - a_2 q \sigma^2 \frac{\partial \theta}{\partial \sigma} \right] \left[1 - a_3 e^{-a_4 q \sigma} \right] \tag{11}$$

The form of (8) remains the same except that ω must now be taken as $\frac{2\pi}{24}$. The boundary conditions (6) and (7) remain identical in form. It will be worth mentioning here that two-meter level may be considered as the lowest level in order to neglect the surface effect.

4. Finite Difference Equations.

In order to obtain a numerical solution of the problem the derivatives of θ with respect to σ in equation (10) and (11) are replaced by appropriate finite difference forms. The problem is then reduced to one of solving a system of linear algebraic equations in the values of θ over a grid of points covering the desired range of time and height. We obtain:

$$\left(\frac{\partial \theta}{\partial \tau}\right)_i = \frac{.36 \times a_1}{q^2 h^2}\left[K\left(\theta_{i+1} - 2\theta_i + \theta_{i-1}\right) + \frac{h}{2}\frac{\partial K}{\partial \sigma}\left(\theta_{i+1} - \theta_{i-1}\right)\right] \tag{12}$$

$$K_i = \left[1 - \frac{a_2}{2} q i^2 h\left(\theta_{i+1} - \theta_{i-1}\right)\right]\left[1 - a_3 e^{-a_4 q i h}\right] \tag{13}$$

$$\left(\frac{\partial K}{\partial \sigma}\right)_i = \left[-a_2 q i^2\left(\theta_{i+1} - 2\theta_i + \theta_{i-1}\right) - a_2 q i\left(\theta_{i+1} - \theta_{i-1}\right)\right]\left[1 - a_3 e^{-a_4 q i h}\right] +$$

$$\left[1 - \frac{a_2}{2} q i^2 h\left(\theta_{i+1} - \theta_{i-1}\right)\right]\left[a_3 a_4 q e^{-a_4 q i h}\right] \tag{14}$$

$$i = 1, 2, \cdots n$$

Here the subscript i designates the ith level of the vertical grid at height ih, where h is the vertical distance between the points of the grid in units of q-meters. Substituting the values of (13) and (14) for i = 1, 2,--- in the right hand side of (12) the respective values of $\left(\frac{\partial \theta}{\partial \tau}\right)_i$ for i = 1, 2,--- are obtained. Now the partial differential equation (1) has now been reduced to a system of ordinary differential equations, which will be integrated numerically by the Runga - Kutta method.

5. Computation.

For this particular problem q is selected as $q = 10^3$. This means the units of σ are in thousands of meters. In order to achieve reasonable accuracy a space interval of ten meters is selected in the vertical. This will be an appropriate time to mention that there will be less accuracy in the results below ten meters. All the figures below this level are approximated linearly. A fairly detailed picture of the vertical structure is obtained assuming that the upper boundary condition (6) applies at the height of 290 meters. Therefore $\theta_{2q} = 0$. Hence the system (12) to be solved consists of 28 simultaneous ordinary differential equations, together with equation (8) for θ_0 . A numerical solution of this system is obtained by the Runga - Kutta method on a National Cash Register 102A electronic computer. The choice of the appropriate time interval over which the integration is to be carried out depends intimately upon the size of the eddy diffusivity. As the latter is increased, $\Delta\tau$ must be decreased in order to maintain sufficient accuracy. With the value of $a_1 = 5 \times 10^5$, the time interval of the integration period was prohibitively small for the type of computer being used, i e., thousands of hours of machine time would be required to complete one full cycle. Hence a smaller value of $a_1 = 1.92 \times 10^3$ was used, which permitted a time interval of 1/16 hours.

6. Results.

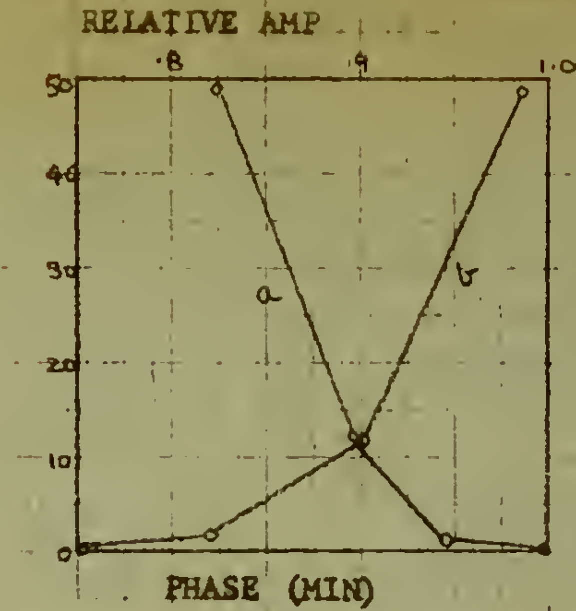
The reduction of a_1 from 5×10^5 to 1.92×10^3 reduces the coefficient of diffusivity which, in turn, reduces the amplitude and increases the phase lag with elevation. In Fig. (3) relative amplitude and phase lag in minutes are plotted against height in meters. It may be observed that the amplitude at ten meters is approximately one half of the surface value and the phase lag at this level is 78 minutes. These values are not uncommon in winter. These values may be compared with those observed at Porton in December on clear days. For instance at 17 meters at Porton the phase lag is 72 minutes and theory gives 63 minutes. In Fig. (3) for comparative purposes, the amplitude reduction and phase lag observed at the University of Washington micrometeorological mast and tower is given in the inset. Notice that in the theoretical results amplitude falls off more rapidly at lower levels where K is small and the change of phase is smaller than at upper levels. The lower values of amplitude and larger phase lags compared with observed data once again indicative of the small value of K. The value of K in the present analysis at the surface is $K = 1.95 \times 10^1$ and at the topmost layer, $K = 1.95 \times 10^3$. These are very small values particularly for clear days in August on the land.

As given in Table (1), the magnitude of lapse rate (in deg per cm) at lower levels, considering above the two meter level, is of the order, 10^{-2}; and between ten and forty meters, this order is of 10^{-3}. In general the lapse rate continues to decrease with elevation.

Fig. 3

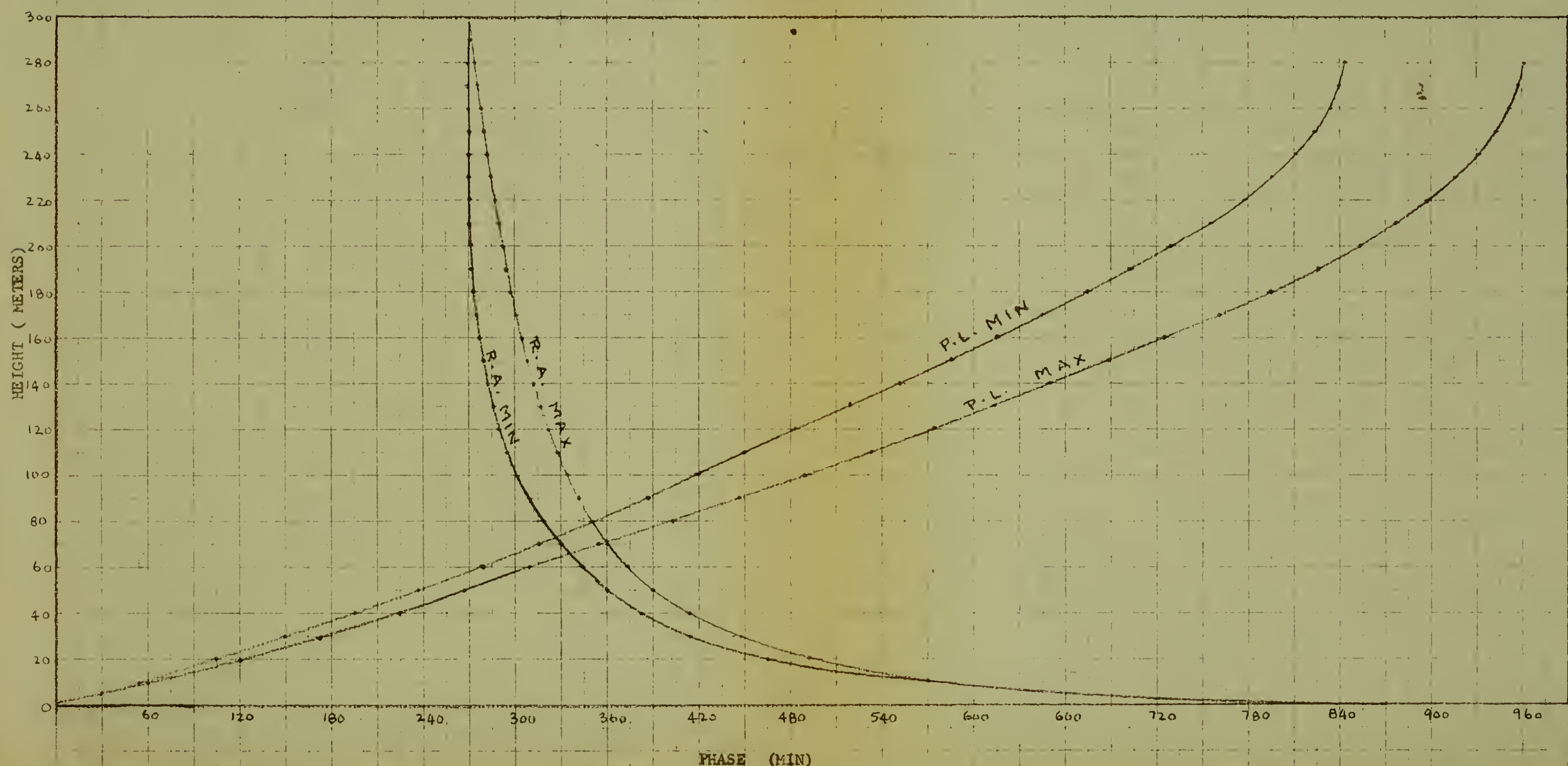
MEAN R. A. AND P. L. OBSERVED AT UNIV OF WASH MICROMETEOROLOGICAL TOWER FOR 23 DAYS OF AUG. 1950

R. A. (a)
P. L. (b)



MEAN RELATIVE AMPLITUDE AND PHASE

RELATIVE AMPLITUDE



From the computation, the values at lower levels (about 10 meters) and upper levels are of order 10^{-4} and 10^{-5} respectively which are about one-hundredth of the observed values. This decrease in lapse rate may be overcome by selecting a_2 in equation (5) hundred times large. In equation (12) instead of reducing a_1 which decreases the magnitude of the coefficient of diffusivity, the factor q may be increased to achieve computational stability in the computer. The increase in q increases the finite vertical distance h over which the devivatives are evaluated. This implies that the accuracy of the finite- difference approximation will decrease.

As an example of this technique, let q be increased by a factor of 10, then K will be equal to 1.95×10^5, which is a reasonable value for the diffusion coefficient. Then the finite vertical distance h will be 100 meters. In this case Fig. (3) may be considered as plotted with height from zero to 3000 meters. This interpretation of the results may be of value if the temperature changes at higher elevations are of primary interest.

Fig. (3) also shows the amplitude reduction and phase lag for the minimum value of the temperature at various elevations. The amplitude reduction for the minimum shows greater reduction than the maximum, which apparently corresponds to the lower nocturnal diffusivity. There is an appreciable difference between the amplitude reduction of the maximum and that of the minimum. For example, at the 40 meters level, the maximum is about 26% of the surface value of θ, while the minimum is about 19% of the surface minimum. At much lower and much higher levels, the difference between amplitude reduction

of the maximum and minimum is less. The phase lag increases with altitude for both maximum and minimum, but the rate of increase decreases with height, because of the increase of the coefficient of diffusivity. The phase lag of the minimum is somewhat less than that of the maximum which does not seem to be consistent with greater amplitude reduction of minima compared to the maxima. Perhaps these differences would vary in succeeding maxima and minima as a greater time elapsed from the initial conditions.

Some comparisons between the observed data and theoretical results may be made by a proper choice of q which indicates the appropriate diffusion coefficient. Table (2) shows the amplitude reduction and phase lag from observation taken at Leafield. The agreement here is fair at elevations above 60 meters. The choice of $q=6$ gives the finite difference interval of 60 meters. Table (3) shows some observation taken at the University of Washington, Seattle. In this case the finite difference interval is 160 meters, but observations are below this height. However, as a first approximation, the amplitude reduction and phase lag are interpolated between 0 and 160 meters and tabulated. The results fit well, but the accuracy of the interpolation is doubtful. Similarly Table (4) compares the theory with observed data at Eiffel Tower. Here q is very large giving rise to finite difference of $h = 600$ meters, whereas the heights considered at the Eiffel Tower are below 300 meters. Yet the comparison with the theory is fair. In Table (5) comparison is made with the results taken at Porton. In this case $h = 20$ meters which is not so great as to offset the results of computation. The theory fits well with

the observed data. Table (6) compares the amplitude reduction in summer and winter at Leafield with theory considering q = 25 and 6 respectively, this gives h = 250 and 60 meters respectively. The values of θ considered here are proportional to the ten-meter level. The magnitude of amplitude reduction in theory and observation are very close to each other.

Table.2.

Amplitude reduction (A.R.) and phase lag (P.L.) taken for the case q = 6 versus observation taken at Leafield, England.

Theory		Leafield		
A.R.	P.L.	Height	A.R.	P.L.
.73	33	25 meters	.90	- hours.
.68	42	30 "	-	1.20 "
-	-	50 "	.84	- "
.52	1.3	60 "	-	1.5 "
.43	1.75	90 "	-	1.66 "
.41	1.8	100 "	.77	- "
.37	2.3	120 "	-	- "

Table.3.

Amplitude reduction and phase lag taken from case q = 16 versus observation at the University of Washington, Seattle.

Theory		Observation		
A.R.	P.L.	Height	A.R.	P.L.
.91	12 min	15 meters	.91	16 min
.83	18 "	30 "	.86	18 "
.79	27 "	45 "	.83	23 "
.73	33 "	60 "	-	- "

Table.4.

Amplitude reduction from Eiffel tower versus theoretical values for q = 60.

Theory	Observation		
A.R.	Height(meters)	Summer	Winter
.79	195	.88	.83
.68	300	.75	.60

Table.5.

Amplitude reduction and phase lag taken for the case q = 2 versus obsertion taken at Porton, England.

Theory		Porton		
A.R.	P.L.	Height	A.R.	P.L.
.75	.57	7 meters	.73	.95 hours
.58	1.05	17 "	.65	1.2 "

Table.6.

Amplitude reduction from Leafield (England) versus theoretical values.

Height in meters	10	25	50	75	100
Relative amplitude.					
(a) June.	1.00	0.90	0.84	0.79	0.77
(b) December.	1.00	0.74	0.54	0.44	0.40
Theory: q = 25	1.00	0.90	0.84	0.78	0.64
q = 6	1.00	0.64	0.58	0.47	0.42

7. Conclusions.

A numerical solution has been presented for the diurnal temperature variation with a coefficient of eddy diffusivity which is a function of height and lapse rate. By suitable selection of several parameters, reasonably good agreement has been obtained between theory and observation.

Improvement in the results may possibly be achieved by improving the functional form of diffusivity coefficient. It would be particularly desirable to make a detailed study of the relationship between eddy diffusivity and such parameters as surface roughness, wind speed, in addition to stability. If a suitable relationship could be obtained for K in terms of these parameters then the solution for the diurnal temperature wave would have a broader application.

8. Acknowledgement.

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9. Bibliography.

(1) O. G. Sutton, Micrometeorology, McGraw-Hill, New York, 333 pp, 1953.

(2) H. Kohler, Ein kurzes Studium des Austausches auf Grund des polenz gesetzes. Beitr. Phys. Fr. Atmos, 19, 91-104, 1932.

(3) B. Haurwitz, The daily temperature period for a linear variation of the Austausch coefficient. Trans. Roy. Soc. Canada, 3rd Sev, 30, 1-8, 1936.

(4) H. Poppendiek, A periodic heat transfer analysis for an atmosphere in which the eddy diffusivity varies sinusoidally with time and linearly with height. J. Meteor. 9, 368-370, 1952.

(5) D. O. Staley, The diurnal temperature wave for eddy conductivity. J. Meteor. 13, 13-20, 1956.

(6) G. J. Haltiner and F. L. Martin, Dynamical and Physical Met, McGraw-Hill, New York, pp 251-263, 1957.

(7) G. J. Haltiner, The diurnal temperature wave with a coefficient of eddy diffusivity which varies exponentially with height and periodic with time. Unpublished manuscript. 1957.